Quasi-Baer module hulls and examples

(Joint work with Jae Keol Park, S. Tariq Rizvi, and Cosmin S. Roman)

Dedicated to S. Tariq Rizvi

Gangyong Lee

Chungnam National University, Assistant Professor

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1946 : Rickart studied C^*-algebras (i.e, Banach algebras with an involution $*$ such that $\|xx^*\| = \|x\|^2$) in which the right annihilator of any element is generated by a projection ($e^* = e, e^2 = e$). (named Rickart C^*-algebras by Kaplansky later.)

1951 : Kaplansky defined AW^*-algebras: C^*-algebras in which the right annihilator of any subset is generated by a projection.

1955 : Kaplansky defined Baer*-rings and Baer rings:
A Baer*-ring (resp. Baer ring) is a *-ring (resp. ring) in which the right annihilator of any subset is generated by a projection (resp. an idempotent).

1960 : Maeda defined Rickart rings (known as p.p. rings)
Also, defined by Kaplansky, Hattori (1960): A ring is called right Rickart if the right annihilator of any single element is generated by an idempotent, equivalently, any principal right ideal is projective.

1967 : Clark defined quasi-Baer rings:
A ring R is called quasi-Baer if the right annihilator of any 2-sided ideal is generated by an idempotent.

One application of quasi-Baer ring hulls of semiprime rings has been that these hulls establish useful connections of quasi-Baer rings to C^*-algebras in Functional Analysis.

Theorem (2009, Birkenmeier, Park, Rizvi)

A unital C^*-algebra R is boundedly centrally closed iff R is a quasi-AW^*-algebra.

$\therefore$ the local multiplier algebra of a C^*-algebra is always a quasi-Baer ring. Consequently, a C^*-algebra whose local multiplier algebra is a C^*-direct product of prime C^*-algebras can be fully characterized.

Let M be a right R-module and $S = \text{End}_R(M)$.

Definition (2004, Rizvi, Roman)

A module M_R is called Baer module if for any left ideal I of S, $r_M(I) = fM$ for some $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$.
Equivalently, a module M_R is Baer if, for any $N_R \leq M_R$, there exists $e^2 = e \in S$ such that $\ell_S(N) = Se$, where $\ell_S(N) = \{f \in S \mid f(N) = 0\}$.

Definition (2007, Rizvi, Roman)

A module M_R is called a Rickart module if for each $\phi \in S$,
$r_M(\phi) = \mathrm{Ker}(\phi) = eM$ for some $e^2 = e \in S$.

Definition (2004, Rizvi, Roman)

A module M_R is called a quasi-Baer module if, for any ideal J of S,
$r_M(J) = fM$ for some $f^2 = f \in S$.
Equiv., M_R is quasi-Baer if, for each fully invariant submodule N of M,
$\ell_S(N) = Se$ for some $e^2 = e \in S$.

It has been of interest to investigate finite dimensional algebras over an arbitrary algebraically closed field.
Clark initially defined a quasi-Baer ring to help characterize a finite dimensional algebra over an algebraically closed field to be a twisted semigroup algebra.

Historically, it is of interest to note that the Hamilton quaternion division algebra over the real number field $\mathbb{R}$ is a twisted group algebra of the Klein four group V_4 over $\mathbb{R}$.

Definition (2013, Birkenmeier, Park, Rizvi)

Let M_R be a module. We fix an injective hull $E(M_R)$ of M_R. Let $\mathfrak{M}$ be a class of modules. We call, when it exists, a module H_R the $\mathfrak{M}$ hull of M_R if H_R is the smallest extension of M_R in $E(M_R)$ that belongs to $\mathfrak{M}$.

Notation We use **qB**$(-)$, **Ric**$(-)$, **B**$(-)$, **Ex**$(-)$, and **FI**$(-)$ to denote the quasi-Baer module hull, the Rickart module hull, the Baer module hull, the extending module hull, and the FI-extending module hull of a module, respectively it they exist.

Definition

For a given module M, the smallest quasi-Baer (resp., Rickart) overmodule of M in $E(M)$ is called the quasi-Baer (resp., Rickart) module hull of M.

Definition (2013, Armendariz, Birkenmeier, Park)

A ring R is called ideal intrinsic over $\mathrm{Cen}(R)$
if $I \bigcap \mathrm{Cen}(R) \neq 0$ for any $0 \neq I \trianglelefteq R$.

1. For a semiprime ring R which is ideal intrinsic over $\mathrm{Cen}(R)$, it is known that R is left (right) nonsingular by [1, Proposition 1.2].

2. If a ring R is semiprime PI, then R is ideal intrinsic over $\mathrm{Cen}(R)$ ([3, Theorem 1.17]).

Recall that a ring R is called a PI-ring if R satisfies a polynomial identity.

Note

(i) If a ring R is semiprime, then the ring $R\mathcal{B}(Q(R))$ is the smallest quasi-Baer intermediate ring between R and $Q(R)$.

(ii) If a ring R is reduced, then $R\mathcal{B}(Q(R))$ is reduced, so $R\mathcal{B}(Q(R))$ is a Baer ring since any reduced quasi-Baer ring is Baer. Therefore, $R\mathcal{B}(Q(R))$ is the smallest Baer ring between R and $Q(R)$, that is, the Baer ring hull of R.

Theorem (2018, Lee, Park, Rizvi, Roman)

Let a ring R be semiprime and ideal intrinsic over $Cen(R)$, n be a positive integer, and $e^2 = e \in End(R_R^{(n)})$. Then $\mathbf{qB}(eR_R^{(n)}) = eR\mathcal{B}(Q(R))_R^{(n)}$.

Therefore, any finitely generated projective module over R has a quasi-Baer hull.

Corollary

Let a ring R be semiprime and ideal intrinsic over $Cen(R)$, and let P_R be a finitely generated projective module over R. Then $\mathbf{qB}(P_R) = \mathbf{FI}(P_R)$.

The following example illustrates that the previous results do not hold for the existence of the Baer hull or the Rickart hull of a finitely generated projective module over a ring R even when R is a commutative domain.

Example

Let R be a commutative domain and n an integer with $n > 1$. Then:

(i) $R_R^{(n)}$ has a Baer hull if and only if R is a Prüfer domain.

(ii) Similarly, $R_R^{(n)}$ has a Rickart hull if and only if R is a Prüfer domain. Hence $(\mathbb{Z}[x] \oplus \mathbb{Z}[x])_{\mathbb{Z}[x]}$ has no Rickart hull.

Recall that a commutative domain R is called Prüfer
if R is semihereditary (i.e., every finitely generated ideal is projective).

Corollary

Assume that A is a Boolean ring and $R = Mat_k(A)$, where k is a positive integer. Let P_R be a finitely generated projective module over R. Then:
(i) P_R has a Baer hull.
(ii) P_R has an extending hull.
(iii) The quasi-Baer hull, the Baer hull, the injective hull, the quasi-injective hull, the continuous hull, the quasi-continuous hull, the extending hull, and the FI-extending hull of P_R all exist and coincide.

Let A be a Boolean ring and $R = \mathrm{Mat}_k(A)$, k a positive integer.
Assume that P_R is a finitely generated projective module over R.
In view of the above corollary, one may expect that $\mathbf{qB}(R_R) = \mathbf{Ric}(P_R)$?

Example

Let $A = \{(a_n) \in \prod_{n=1}^{\infty} \mathbb{Z}_2 \mid a_n \text{ is eventually constant}\}$. Then A is a Boolean ring. Put $R = \mathrm{Mat}_k(A)$, where k is any positive integer.
We note that $Q(\mathrm{Mat}_k(A)) = \mathrm{Mat}_k(Q(A))$ and $Q(A) = \prod_{n=1}^{\infty} \mathbb{Z}_2$.
$\therefore \mathbf{qB}(R_R) = \mathbf{B}(R_R) = \mathbf{Ex}(R_R) = \mathbf{FI}(R_R) = E(R_R) = \mathrm{Mat}_k(\prod_{n=1}^{\infty} \mathbb{Z}_2)$.
Since A is a Boolean ring, R is von Neumann regular, so R_R is Rickart.
Thus $\mathbf{Ric}(R_R) = R_R \neq E(R_R)$. Therefore $\mathbf{qB}(R_R) \neq \mathbf{Ric}(R_R)$

Lemma

Let R be a Dedekind domain which is not a field. Assume that M is an R-module such that $Ann_R(M) \neq 0$, and $\{K_i \mid i \in \Lambda\}$ is a set of nonzero submodules of F_R, where F is the field of fractions of R.
Put $N_R = M_R \oplus (\bigoplus_{i\in\Lambda} K_i)_R$. Then we have the following.
(i) If N_R has a quasi-Baer or a Rickart essential extension, then M_R is semisimple.
(ii) $M_R \oplus E[(\bigoplus_{i\in\Lambda} K_i)_R]$ is a (quasi-)Baer module if and only if $M_R \oplus E[(\bigoplus_{i\in\Lambda} K_i)_R]$ is a Rickart module if and only if M_R is semisimple.

Theorem (2018, Lee, Park, Rizvi, Roman)

Let R be a Dedekind domain. Assume that M is an R-module such that $I := Ann_R(M) \neq 0$, and $\{K_i \mid i \in \Lambda\}$ is a set of nonzero submodules of F_R, where F is the field of fractions of R. Then the following are equivalent.
(i) $M_R \oplus (\bigoplus_{i\in\Lambda} K_i)_R$ has a quasi-Baer hull.
(ii) M_R is semisimple.
In this case, $\mathbf{qB}(M_R \oplus (\bigoplus_{i\in\Lambda} K_i)_R) = M_R \oplus (\bigoplus_{i\in\Lambda} K_i T(I))_R$, where $T(I)$ is the Nagata transform of I. Further, $T(I) = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{k=1}^{n} a_k q_k$ with $a_k \in I$ and $q_k \in I^{-1}$, $1 \leq k \leq n$.

Assume that R is a commutative domain with the field of fractions F.
Let B be a nonzero ideal of R. We put $B^0 = R$.
For each $0 \leq \ell$, let $[R : B^\ell] = (B^\ell)^{-1} = \{q \in F \mid qB^\ell \subset R\}$.
We take $T(B) = \bigcup_{\ell \geq 0} [R : B^\ell]$. Then

$$T(B) = \sum_{\ell \geq 0} [R : B^\ell] = \sum_{\ell \geq 0} (B^\ell)^{-1}$$

since $R = [R : B^0] \subseteq [R : B] \subseteq [R : B^2] \subseteq \dots$.
$T(B)$ is an intermediate domain between R and the field of fractions of R.
$T(B)$ is called the Nagata transform (or ideal transform) of B
(see [13, p.490] and [15, p.325]).
For an invertible ideal I of R, let $I^{-2} = I^{-1}I^{-1}$, $I^{-3} = I^{-1}I^{-1}I^{-1}$, and so on.

Corollary

Let R be a Dedekind domain. Assume that N is an R-module with $N/t(N)$ projective and $Ann_R(t(N)) \neq 0$.
Then the following are equivalent.
(i) N has a quasi-Baer hull.
(ii) $t(N)$ is semisimple.

Let R be a semiprime PI-ring and P_R be a finitely generated projective module. Then $\mathbf{qB}(P_R) = \mathbf{FI}(P_R)$ from the previous result.

However, these two hulls do not coincide for the case of finitely generated modules over $\mathbb{Z}$.

Example

Let $N = \mathbb{Z}_p \oplus \mathbb{Z}$, where p is a prime integer.
Then $\mathbf{FI}(N) = N$ because N itself is an FI-extending $\mathbb{Z}$-module.
However, $\mathbf{qB}(N) = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$.
So N is finitely generated, but $\mathbf{qB}(N) \neq \mathbf{FI}(N)$.

Theorem (2018, Lee, Park, Rizvi, Roman)

Let R be a Dedekind domain.
Assume that M is an R-module with $I := Ann_R(M) \neq 0$,
and let $\{K_i \mid i \in \Lambda\}$ be a set of nonzero fractional ideals of R.
We put $N_R = M_R \oplus (\bigoplus_{i \in \Lambda} K_i)_R$. Then the following are equivalent.
(i) N has a quasi-Baer hull.
(ii) N has a Rickart hull.
(iii) M is semisimple.
(iv) $M_R \oplus E[(\bigoplus_{i \in \Lambda} K_i)_R]$ is a Baer module.
In this case, $\mathbf{qB}(N_R) = \mathbf{Ric}(N_R) = M_R \oplus (\bigoplus_{i \in \Lambda} K_i T(I))_R$,
where $T(I)$ is the Nagata transform of I.
Further, $T(I) = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{k=1}^{n} a_k q_k$ with $a_k \in I$ and $q_k \in I^{-1}$, $1 \leq k \leq n$.

Theorem (2018, Lee, Park, Rizvi, Roman)

Let R be a Dedekind domain. Assume that N is an R-module with $N/t(N)$ projective and $Ann_R(t(N)) \neq 0$.
Then the following are equivalent.
(i) N has a quasi-Baer hull.
(ii) N has a Rickart hull.
(iii) $t(N)$ is semisimple.
(iv) $t(N) \oplus E(N/t(N))$ is a Baer module.
In this case, $\mathbf{qB}(N_R) = \mathbf{Ric}(N_R) \cong t(N) \bigoplus (N/t(N))T(I) \cong (\bigoplus_{i\in\Gamma} R/P_i)_R \oplus (\bigoplus_{i\in\Lambda} K_i T(I))_R$,
where $T(I)$ is the Nagata transform of $I := Ann_R(t(N))$.
Further, $T(I) = R[q_1, q_2, \ldots, q_n]$,
where $1 = \sum_{k=1}^{n} a_k q_k$ with $a_k \in I$ and $q_k \in I^{-1}$, $1 \leq k \leq n$.

Corollary

Let R be a commutative PID.
Assume that M is an R-module with $Ann_R(M) \neq 0$*, and* Λ *is any set.*
Put $N = M_R \oplus R_R^{(\Lambda)}$*. Then the following are equivalent.*
(i) N has a quasi-Baer hull.
(ii) N has a Rickart hull.
(iii) M is semisimple.
(iv) $M \oplus E(N/t(N))$ *is a Baer module.*
In this case, $\mathbf{qB}(N_R) = \mathbf{Ric}(N_R) = M_R \oplus R[1/a]_R^{(\Lambda)}$*,*
where $Ann_R(M) = aR$*.*

Corollary

Let R be a Dedekind domain. Assume that N is an R-module with $N/t(N)$ finitely generated and $Ann_R(t(N)) \neq 0$.
Then the following are equivalent.
(i) N has a quasi-Baer hull.
(ii) N has a Rickart hull
(iii) N has a Baer hull.
(iv) $t(N)$ is semisimple.
(v) $t(N) \oplus E(N/t(N))$ is a Baer module.
In this case, $\mathbf{qB}(N) = \mathbf{Ric}(N) = \mathbf{B}(N)$.

The following example illustrates the previous results.

Example

Let Γ_i, $i = 1, 2, 3$, are nonempty sets, and let $M = \mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)}$.
(i) For any positive integer m, let $V_m = M \oplus \mathbb{Z}^{(m)}$. Then
$\mathbf{qB}(V_m) = \mathbf{Ric}(V_m) = \mathbf{B}(N) = M \oplus \mathbb{Z}[1/30]^{(m)}$ as $\mathrm{Ann}_{\mathbb{Z}}(M) = 30\mathbb{Z}$.

(ii) For any nonempty set Ω, let $N_\Omega = M \oplus \mathbb{Z}^{(\Omega)}$.
Then $\mathbf{qB}(N_\Omega) = \mathbf{Ric}(N_\Omega) = M \oplus \mathbb{Z}[1/30]^{(\Omega)}$ as $\mathrm{Ann}_{\mathbb{Z}}(M) = 30\mathbb{Z}$.

Example

Assume that $M = \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i}$, where n is a positive integer, and all p_i are prime integers. Say $p_1, p_2, \ldots, p_s$ are all the distinct prime integers in $\{p_1, p_2, \ldots, p_n\}$. Let $a = p_1 p_2 \cdots p_s$.

Then there exists a set Λ (necessarily infinite) such that:

(i) $M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$ is not a Baer $\mathbb{Z}$-module.

(ii) $M \oplus \mathbb{Z}^{(\Lambda)}$ has no Baer hull.

In contrast to (i) and (ii), we have the following.

(iii) $\mathbf{qB}(M \oplus \mathbb{Z}^{(\Lambda)}) = \mathbf{Ric}(M \oplus \mathbb{Z}^{(\Lambda)}) = M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$.

Furthermore, the quasi-Baer (resp., Rickart) module hull of a direct sum of two modules is not isomorphic to the direct sum of their quasi-Baer (resp., Rickart) module hulls (if each hull exists).

(iv) $\mathbf{qB}(M \oplus \mathbb{Z}^{(\Lambda)}) \ncong \mathbf{qB}(M) \oplus \mathbf{qB}(\mathbb{Z}^{(\Lambda)})$
and $\mathbf{Ric}(M \oplus \mathbb{Z}^{(\Lambda)}) \ncong \mathbf{Ric}(M) \oplus \mathbf{Ric}(\mathbb{Z}^{(\Lambda)})$.

Theorem

Let R be a Dedekind domain and N be a finitely generated R-module. Then the following are equivalent.
(i) N is quasi-Baer.
(ii) N is Rickart.
(iii) N is Baer.
(iv) N is semisimple or torsion-free.

Theorem

Let R be a Dedekind domain and N be a direct sum of finitely generated R-modules. Then the following are equivalent.
(i) N is quasi-Baer.
(ii) N is Rickart.
(iii) N is semisimple or torsion-free.

Thank you

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